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Parallel Transport Frame of Smarandache Curves in Euclidean Space

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Abstract: In the present work, we introduce Parallel transport frames of Smarandache curves in Euclidean space. In the first section, we give the basic tools of a parallel transport frame of a curve in 4-dimensional Euclidean space. In the second section, we study Smarandache curve of Euclidean space in parallel transport frame, we solve a few theorems, corollaries and examples. Again third section, we define parallel transport frame to the Smarandache curve and obtain some definitions and their apparatus. Further fourth section, we have also explained to Frenet frame of principal normal, binomial and their derivatives in the curvatures of the curve. In the end section, we discussed about the Smarandache curve in the Euclidean space of all apparatus Frenet-Serret in the differential geometry.
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Key Words: Smarandache Curves • Frenet-Serret • Euclidean Space • Parallel transport frame

1. Introduction

Let $\beta : I \subset R \to E^4$ be in E^4. $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$, and $Z = (z_1, z_2, z_3, z_4)$, where X, Y, Z be any three vectors in E^4. Where β is parameterized by arc length of s if $< \beta'(s), \beta'(s) >= 1$, together with the inner product of E^4 given by

(1.1) $< X, Y >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$

$X \in E^4$ is

$$\|X\| = \sqrt{< X, X >} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

$$\|X\| = \left(\sum_{i=1}^{4} |x_4|^p \right)^{\frac{1}{p}}, \quad p = 1 \quad \Rightarrow \quad \|x_4\|_\infty = \max|x_4|$$

$$\Rightarrow \quad \frac{\partial}{\partial x_k}\|x\|_p = \frac{x_k|x_k|^{p-2}}{\|x\|_p^{\ p-1}} \quad \Rightarrow \quad \frac{\partial}{\partial x}\|x\|_p = \frac{x \bullet |x_k|^{p-2}}{\|x\|_p^{\ p-1}}$$

If $p = 2,$ then $$\frac{\partial}{\partial x}\|x\|_2 = \frac{x}{\|x\|_2}$$

Then their vector product is defined by the determinant

$$(1.2) \quad X \times Y \times Z = \begin{vmatrix} a & b & c & d \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

Suppose (t, m_1, m_2, m_3) be the working Frenet frame of curve β. Then t, n, b_1 and b_2 are the tangent, the principal normal, and 1st and 2nd binomial vectors of the curve β, respectively. Then Frenet-Serret frame is given by

$$(1.3) \quad \begin{bmatrix} t' \\ n' \\ b_1' \\ b_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_1 \end{bmatrix}$$

Where $<t,t>=<n,n>=<b_1,b_1>=<b_2,b_2>=1,$

$<t,n>=<t,b_1>=<n,b_1>=<t,b_2>=<n,b_2>=0$

Here κ, τ, σ denote principal curvature to the Serret-frenet of the curve β. While t', n', b_1' and b_2' are called tangent, principal normal and 1st and 2nd binomial.

Let $\beta = \beta(t)$ be in E^4. The curve β can be solved by

$$t = \frac{\beta'}{\|\beta'\|}, \qquad n = \|\beta'\|^2 \beta'' - <\beta', \beta''> \beta', \qquad b_1 = \rho(b_2 \times t \times n)$$

$$b_2 = \rho \left\{ \frac{t \times n \times \beta'''}{\|t \times n \times \beta'''\|} \right\}, \qquad \kappa = \frac{\| \|\beta'\| \beta'' - <\beta', \beta''> \|}{\|\beta'\|^4}$$

$$\tau = \frac{\|t \times n \times \beta'''\| \|\beta'\|}{\| \|\beta'\|^2 \beta'' - <\beta', \beta''> \beta' \|}, \qquad \sigma = \frac{<\beta'''', b_2>}{\|t \times n \times \beta'''\| \|\beta'\|}.$$

Where ρ is taken ± 1 such that $[t, n, b_1, b_2] = 1$.

We use $T(s)$ and vector fields m_1, m_2 and m_3 to construct an alternative frame. The reason for the name parallel transport frame is because the normal component of the derivatives of the field is zero. If the set (t, m_1, m_2, m_3) as parallel transport frame and

$$k_1 = <t', m_1>, \qquad k_2 = <t', m_2>, \qquad k_3 = <t', m_3>$$

Parallel transport curvatures.

Using Euler angles an arbitrary rotation matrix is given by

$$\begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\phi + \sin\phi\sin\theta\sin\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

, Where angels θ, ϕ, ψ are Euler angles.

2. Smarandache curve of Euclidean Space in parallel transport frame

We define the parallel transport frame of a Smarandache curve and use Euler angles to define the relationships between the frame and the Frenet frame of the Smarandache curve in Euclidean space. For the first time in 4-dimensional Euclidean space, a well-known relation in Euclidean space is generalized.

Theorem 2.1. Let (t, m_1, m_2, m_3) be curve $\beta : I \subset R \to E^4$ and (t, m_1, m_2, m_3) denotes the curve β. The relation may be expressed as the arc is given by

$$t = t(s),$$
$$n = m_1(\cos\lambda\cos\mu) + m_2(-\cos\eta\sin\mu + \sin\eta\sin\lambda\cos\mu) + m_3(\sin\eta\sin\mu + \cos\eta\sin\lambda\cos\mu)$$
$$b_1 = m_1(\cos\lambda\sin\mu) + m_2(\cos\eta\cos\mu + \sin\lambda\sin\eta\sin\mu) + m_3(-\sin\eta\cos\mu + \sin\lambda\cos\eta\sin\mu)$$
$$b_2 = m_1(-\sin\lambda) + m_2(\sin\eta\cos\lambda) + m_3(\cos\eta\cos\lambda)$$

Then frame E^4 are given by

$$(2.1)\quad \begin{bmatrix} t' \\ m_1' \\ m_2' \\ m_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

where k_1, k_2 and k_3 are curve β as follows

$$k_1 = \kappa\cos\lambda\cos\eta, \qquad k_2 = \kappa(-\cos\eta\sin\mu + \sin\lambda\sin\eta\cos\mu),$$
$$k_3 = \kappa(\sin\eta\sin\mu + \sin\lambda\cos\eta\cos\mu) \quad \text{and} \quad \kappa = \sqrt{k_1^2 + k_2^2 + k_3^2},$$
$$\tau = -\psi' + \phi'\sin\lambda, \qquad \sigma = \frac{\lambda'}{\sin\mu}, \qquad \eta\cos\lambda + \lambda'\cot\mu = 0.$$

Where $\lambda' = \dfrac{\sigma}{\sqrt{\kappa^2 + \tau^2}}$, $\quad \mu' = -\tau - \sigma\dfrac{\sqrt{\sigma^2 - \lambda'^2}}{\sqrt{\kappa^2 + \tau^2}}$, $\quad \eta' = -\dfrac{\sqrt{\sigma^2 - \lambda'^2}}{\cos\lambda}$.

Proof: Given that the above theorem, differentiating the m_1, m_2, m_3 with respect to s, we get

$$m_1' = (-\kappa\cos\lambda\cos\mu)t + (\lambda'\sin\lambda\cos\mu - \mu'\cos\lambda\sin\mu - \tau\cos\lambda\sin\mu)n + (\lambda'\sin\lambda\cos\mu - \psi'\cos\theta\sin\psi - \tau\cos\theta\sin\psi)n + (-\theta'\sin\theta\sin\mu - \mu'\cos\lambda\cos\mu - \sigma\sin\lambda)b_1 + (-\lambda'\cos\lambda - \sigma\cos\lambda\sin\mu)b_2,$$

$$m_2' = -\kappa[-\cos\eta\sin\mu + \sin\lambda\sin\eta\sin\mu]t + [\eta'\sin\eta\sin\mu - \mu'\cos\lambda\cos\eta + \eta'\sin\lambda\cos\eta\cos\mu + \lambda'\sin\eta\cos\lambda\cos\mu - \mu'\sin\lambda\sin\eta si\mu - k_2(\cos\eta\cos\mu + \sin\lambda\sin\eta\sin\mu)]n + [-\lambda'\sin\eta\cos\mu - \mu'\cos\mu\sin\mu + \eta'\sin\lambda\cos\eta\sin\mu + \lambda'\cos\lambda\sin\eta\sin\mu + \mu'\sin\lambda\sin\eta\cos\mu + \tau(-\cos\eta\sin\mu + \sin\lambda\sin\eta\cos\mu)$$

$$-\sigma\cos\lambda\sin\eta]b_1+[\eta'\cos\lambda\cos\eta-\lambda'\sin\lambda\sin\eta$$
$$+\sigma(\cos\eta\sin\mu+\sin\lambda\sin\eta\sin\mu]b_2$$
$$m_3'=[-\kappa(\sin\eta\sin\mu+\sin\lambda\cos\eta\cos\mu)]t+[(\eta'\cos\lambda\sin\mu+\mu'\sin\eta\cos\mu$$
$$-\eta'\sin\lambda\sin\eta\cos\mu+\lambda'\cos\lambda\cos\eta\cos\mu-\mu'\sin\lambda\cos\eta\sin\mu-\tau(\sin\eta\cos\mu$$
$$+\sin\lambda\cos\eta\sin\mu)]n+[(-\eta'\cos\eta\cos\mu+\mu'\sin\eta\sin\mu-\eta'\sin\lambda\sin\eta\sin\mu$$
$$+\lambda'\cos\lambda\cos\eta\sin\mu+\mu'\sin\lambda\cos\eta\cos\mu+\tau(\sin\eta\sin\mu+\sin\lambda\cos\eta\cos\mu)$$
$$+\sigma(\cos\lambda\cos\eta)]b_1+(-\lambda'\cos\lambda\sin\eta-\lambda'\sin\lambda\cos\eta)$$
$$+\sigma(-\sin\eta\cos\varpi+\sin\lambda\cos\eta\sin\mu)]b_2.$$

Since m_1, m_2 and m_3 are vector field, normal component of the m_1', m_2' and m_3' must be zero and the equalities are satisfy

$$<m_1',m_2>=<m_1',m_3>=<m_2',m_1>=<m_2',m_3>=<m_3',m_1>=<m_3',m_2>=0$$

Also, if we consider that the parallel transport frame of the curve β we can easily complete proof that the above theorem.

Corollary 2.1. If we consider $\lambda=\eta=\mu=0$ that then transport in E^4. We introduce the relations between the frame and the Serret-Frenet frame of a Smarandache curve in Euclidean space using Euler angles and give the parallel transport frame of a Smarandache curve. For the first time in 4-dimensional Euclidean space, the well-known Euclidean space relation is generalized.

Example 2.1. Let $\beta(s)=(\sin s, 2s+1, 2s-1, s)$ and $\beta''(0)=(0,0,0,0)$ not calculate the Frenet at $s=0.$

$$m_1=b_1\sin\mu, \qquad m_2=b_1\cos\eta\cos\mu, \qquad m_3=-b_1\sin\eta\cos\mu,$$

Where η and μ are constant angles.

Theorem 2.2. Suppose $\beta: I\subset R\to E^4$ be $k_i(i=1,2,3)$ in Euclidean space, then lie-down β if $ak_1+bk_2+ck_3+1=0$

Proof: Let β recline on a sphere where center is P and radius is R, then

$$<\beta-P,\beta-P>=R^2 \qquad \text{it gives us}$$
$$<t,\beta-P>=0, \qquad \Rightarrow \qquad \beta-P=am_1+bm_2+cm_3$$

For $a'=<\beta-P,m_1>=<t,m_1>+<k_1t,\beta-P>=0$

$<t,\beta-P>$, we get

$$<k_1m_1+k_2m_2+k_3m_3,\beta-P>+<t,t>=ak_1+bk_2+ck_3+1=0.$$

That is, $ak_1+bk_2+ck_3+1=0$

Moreover, $R^2=<\beta-P,\beta-P>=a^2+b^2+c^2=\frac{1}{d^2}d$

, the plane $ax+by+cz+1=0$

Conversely, holds

$ak_1 + bk_2 + ck_3 + 1 = 0, \qquad P = \beta - am_1 - bm_2 - cm_3, \qquad P' = t + (ak_1 + bk_2 + ck_3)t = 0$

Similarly shows that $R^2 =< \beta - P, \beta - P >$ is constant.

Example 2.2. Let $\beta(s) = \left(\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s \right)$ be a curve in Euclidean space E^4. there are several formulas for showing that this curve is a spherical curve. But the formulas have some disadvantages which were define the above chapters. Then we calculate curvature functions of the curve β according to parallel transport frame.

$$k_1 = 0,\ k_2 = -\cos\phi\ k_3 = \sin\phi$$

, where ϕ is constant. The curve β satisfy the fallowing equation

$$ak_1 + bk_2 + ck_3 + 1 = 0.$$

Consequently, the curve β is a spherical curve. But, using the Serret-Frenet curvatures we cannot show that β is a spherical curve. Because has a zero torsion.

3. Parallel Transport frame in tb_1 and tm_1 to the Smarandache Curve in E^4

tb_1 and tm_1 Smarandache Curve to parallel transport frames in E^4 then

3.1 tb_1 Smarandache Curves with Parallel Transport Frame

tb_1 Smarandache curves to the parallel transport frame , then obtain their Frenet apparatus.

Definition 3.1. Smarandache curve is a normal in four-dimensional Euclidean space whose location vector is determined by frame vectors.

Definition 3.2. Suppose $\beta = \beta(s)$ be curve and nonzero curvatures k_1, k_2, k_3 and (t, n, b_1, b_1) be working frame on it tb_1 are $\alpha(S_\alpha) = \frac{1}{\sqrt{2}}[t(s) + b_1(s)]$.

Theorem 3.1. Suppose $\beta(s)$, curvatures k_1, k_2, k_3 and $\alpha(s_\alpha)$ be tb_1 Smarandache curves in the frame defined by $\beta(s)$. Serret-Frenet apparatus of $\alpha(t_\alpha, n_\alpha, b_{1\alpha}, b_{2\alpha}, k_{1\alpha}, k_{2\alpha}, k_{3\alpha})$ could be of $\beta(t, n, b_1, b_2, k_1, k_2, k_3)$.

Proof: Let $\alpha = \alpha(s_\alpha)$ be tb_1 of th curve β. Then

By using (2), we get

(3.1) $\alpha(S_\alpha) = \frac{1}{\sqrt{2}}[t(s) + b_1(s)]$

By differentiating (3.1)

(3.2) $\frac{d\alpha(s_\alpha)}{ds} = \frac{d\alpha(s_\alpha)}{ds_\alpha}.\frac{ds_\alpha}{ds} = \frac{1}{\sqrt{2}}[(k_1 - k_2)n + k_3 b_2],$ Then α is

(3.3) $t_\alpha = A_1 n + A_2 b_2$

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Where $\frac{ds_\alpha}{ds} = \sqrt{\frac{(k_1 - k_2)^2 + k_3^2}{2}}$, $A_1 = \frac{k_1 - k_2}{\sqrt{(k_1 - k_2)^2 + k_3^2}}$ and $A_2 = \frac{k_3}{\sqrt{(k_1 - k_2)^2 + k_3^2}}$

Again differentiating the tangent vector of the curve α with respect to s_α, we can get α'' as follows

(3.4) $$\alpha'' = \frac{\sqrt{2}[-k_1(k_1 - k_2)t + (k_1k_2 - k_2^2 - k_3^2)b_1]}{(k_1 - k_2)^2 + k_3^2}$$

The principal normal of the curve α is

(3.5) $$n_\alpha = A_3 t + A_4 b_1$$

Where $A_3 = \frac{-k_1(k_1 - k_2)}{\sqrt{k_1^2(k_1 - k_2)^2 + (k_1k_2 - k_2^2 - k_3^2)^2}}$ and $A_4 = \frac{k_1k_2 - k_2^2 - k_3^2}{\sqrt{k_1^2(k_1 - k_2)^2 + (k_1k_2 - k_2^2 - k_3^2)^2}}$

(3.6) $$\alpha''' = A_5 n + A_6 b_2$$

Where $A_5 = \frac{2[-k_1^2(k_1 - k_2) - k_2(k_1k_2 - k_2^2 - k_3^2)]}{\left[(k_1 - k_2)^2 + k_3^2\right]^{\frac{3}{2}}}$ and $A_6 = \frac{k_3(k_1k_2 - k_2^2 - k_3^2)}{\left[(k_1 - k_2)^2 + k_3^2\right]^{\frac{3}{2}}}$

The 2nd 1st binomial vector of the curve α is

(3.7) $$b_{2\alpha} = \frac{(k_1k_2 - k_2^2 - k_3^2)t + k_1(k_1 - k_2)}{\sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}}$$

(3.8) $$b_{1\alpha} = \frac{-k_3 n + (k_1 - k_2)b_2}{\sqrt{k_3^2 + (k_1 - k_2)^2}}$$

The 1st, 2nd and 3rd curvature of the curve α are

(3.9) $$k_{1\alpha} = \frac{2\left[(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2\right]}{\left[k_3^2 + (k_1 - k_2)^2\right]}$$

(3.10) $$k_{2\alpha} = \frac{\sqrt{2}k_3\left[k_1(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)\right]}{\left[(k_3^2 + (k_1 - k_2)^2\right] + \sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}}$$

(3.11) $$k_{3\alpha} = \frac{\sqrt{2}(-k_1A_4A_5 - k_2A_3A_5 - k_3A_3A_6)}{\sqrt{(k_3^2 + (k_1 - k_2)^2} + \sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}}$$

4. TM_1 Smarandache curves in E^4 according to the parallel transport frame

TM_1 Smarandache curves is curve to the parallel transport frame.

Definition 4.1. Let $\beta = \beta(s)$ be in E^4 and $\{T_\beta, M_{1\beta}, M_{2\beta}, M_{3\beta}\}$ be its working frame. TM_1 Smarandache curves is

(4.1) $$\alpha(s_\alpha) = \frac{1}{\sqrt{2}}(T_\beta + M_{1\beta}).$$

Theorem 4.1. Suppose $\beta=\beta(s)$ be curvatures $K_{1\beta},K_{2\beta},K_{3\beta}$ and $\alpha(s_\alpha)$ be TM_1 in E^4 defined by $\beta=\beta(s)$. Then α can be β and $\alpha(K_{1\alpha},K_{2\alpha},K_{3\alpha})$ can be obtained by β.

Proof: To calculate the TM_1 to $\beta(s)$ differentiating equation (4.1) with respect to s then

(4.2) $$T_\alpha \frac{ds_\alpha}{ds}=\frac{1}{\sqrt{2}}\left(-K_{1\beta}T_\beta+K_{1\beta}M_{1\beta}+K_{2\beta}M_{2\beta}+K_{3\beta}M_{3\beta}\right)$$

The tangent vector of the curve β can be written as

(4.3) $$T_\alpha=\left(\frac{-K_{1\beta}T_\beta+K_{1\beta}M_{1\beta}+K_{2\beta}M_{2\beta}+K_{3\beta}M_{3\beta}}{\sqrt{K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2}}\right)$$

Where $\frac{ds_\alpha}{ds}=\frac{1}{\sqrt{2}}\sqrt{K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2}$

Differentiating (5.3) with respect to s then

(4.4) $$T'_\alpha=\frac{dT_\alpha}{ds_\alpha}=\lambda_0T_\beta+\lambda_1M_{1\beta}+\lambda_2M_{2\beta}+\lambda_3M_{3\beta}$$

Where $\lambda_0=\frac{-\sqrt{2}(K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2)}{(2K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2)}$, $\lambda_1=\frac{-\sqrt{2}K_{1\beta}^2}{(2K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2)}$,

$\lambda_2=\frac{-\sqrt{2}(K_{1\beta}^2K_{2\beta}^2)}{(2K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2)}$, $\lambda_3=\frac{-\sqrt{2}(K_{1\beta}^2K_{3\beta}^2)}{(2K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2)}$

Then the curve α is

(4.5) $$k_{1\alpha}=\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_2^3}=\frac{\sqrt{2}\sqrt{K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2}}{\sqrt{2K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2}}$$

The α is

(4.6) $$n_\alpha=\frac{\lambda_0T_\beta+\lambda_1M_{1\beta}+\lambda_2M_{2\beta}+\lambda_3M_{3\beta}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}}$$

The 3rd derivative of α is

(4.7) $$\alpha'''=(\lambda_0T'_\beta+\lambda_1M'_{1\beta}+\lambda_2M'_{2\beta}+\lambda_3M'_{3\beta})\frac{\sqrt{2}}{\sqrt{2K_{1\beta}^2+K_{2\beta}^2+K_{3\beta}^2}}$$

$$\Rightarrow T_\alpha\times n_\alpha\times\alpha'''=C_1M_{1\beta}+C_2M_{2\beta}+C_3M_{3\beta}$$

Where $$C_1=\frac{\sqrt{2}\left[\lambda_0\lambda_3K_{1\beta}K_{2\beta}-\lambda_0\lambda_2K_{1\beta}K_{3\beta}-\left(\lambda_1K_{1\beta}+\lambda_2K_{2\beta}+\lambda_3K_{3\beta}\right)\left(\lambda_3K_{2\beta}-\lambda_2K_{3\phi}\right)\right]}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\left(2K_{1\beta}^2+K_{2\beta}^2+K_{3\phi}^2\right)}$$

$$C_2=\frac{\sqrt{2}\left[\lambda_0\lambda_3K_{1\beta}^2-\lambda_0\lambda_1K_{1\beta}K_{3\beta}-\left(\lambda_1K_{1\beta}+\lambda_2K_{2\beta}+\lambda_3K_{3\beta}\right)\left(\lambda_3K_{1\beta}-\lambda_1K_{3\beta}\right)\right]}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\left(2K_1^2+K_2^2+K_3^2\right)}$$

$$C_3 = \frac{\sqrt{2}\left[\lambda_0\lambda_2 K_{1\beta}^2 - \lambda_0\lambda_1 K_{1\beta}K_{2\beta} - \left(\lambda_1 K_{1\beta} + \lambda_2 K_{2\beta} + \lambda_3 K_{3\beta}\right)\left(\lambda_1 - \lambda_2\right)K_{2\beta}\right]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left(2K_1^2 + K_2^2 + K_3^2\right)}$$

We know the 2nd and 1st binomial of the curve α is

(4.8) $$b_{2\alpha} = \frac{C_1 M_{1\beta} + C_2 M_{2\beta} + C_3 M_{3\beta}}{\sqrt{C_1^2 + C_2^2 + C_3^2}}$$

(4.9) $$b_{1\alpha} = b_{2\alpha} \times T_\alpha \times n_\alpha = \gamma_0 T_\beta + \gamma_1 M_{1\beta} + \gamma_2 M_{2\beta} + \gamma_3 M_{3\beta}$$

Where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are constants and $M_{1\alpha}$, $M_{2\alpha}$, and $M_{3\alpha}$ are frame of $M_{1\beta}$, $M_{2\beta}$, and $M_{3\beta}$ of the curve α is given. Further 2nd and 3rd curvature tensor of the curve α is

(4.10) $$k_{2\alpha} = \sqrt{\frac{C_1^2 + C_2^2 + C_3^2}{\sqrt{\lambda_0^1 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}}}$$

(4.11) $$k_{3\alpha} = \frac{-(C_1 K_{1\beta} + C_2 K_{2\beta} + C_3 K_{3\beta})(\lambda_1 K_{1\beta} + \lambda_2 K_{2\beta} + \lambda_3 K_{3\beta})}{\sqrt{C_1^2 + C_2^2 + C_3^2}}$$

Then 1st , 2nd and 3rd curvature of the curve α is

(4.12) $$K_{1\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\cos\theta_\alpha \cos\psi_\alpha$$

(4.13) $$K_{2\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left[-\cos\phi_\alpha \sin\psi_\alpha + \sin\phi_\alpha \sin\theta_\alpha \cos\psi_\alpha\right]$$

(4.14) $$K_{3\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left[\sin\phi_\alpha \sin\psi_\alpha + \cos\phi_\alpha \sin\theta_\alpha \cos\psi_\alpha\right]$$

Discussion

We discussed an Einstein's theory opened a door to new geometries such as Smarandache curve of parallel transport frame in the Euclidean space. They adapted the geometrical models to relativistic motion of charged particles. The dynamics of charged particles are currently defined by the Frenet-Serret formalism of relativistic motion. On another usual curve, a Smarandache is a position vector made up of Frenet frame vectors. In this paper, we investigate a unique Curve in Euclidean space.

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